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Microfunctions and a transfer operator for complex dynamical systems.

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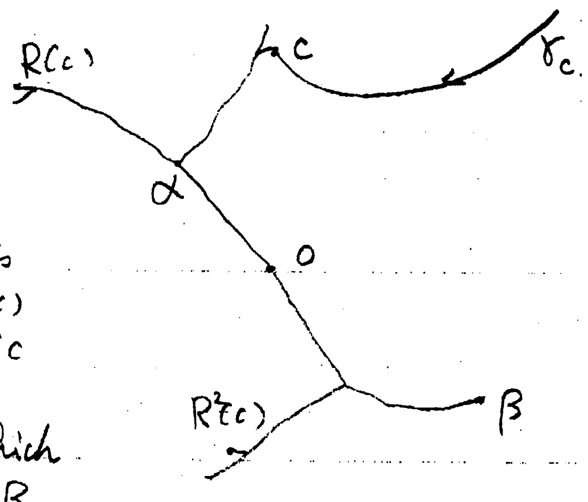
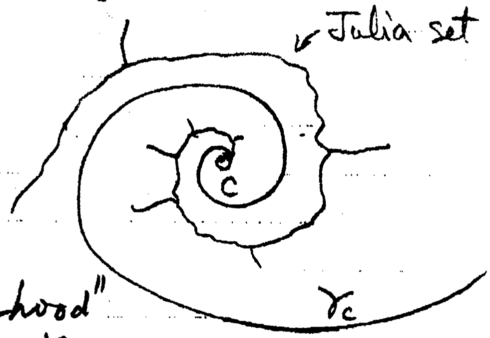
§1. Functions with regular singularities

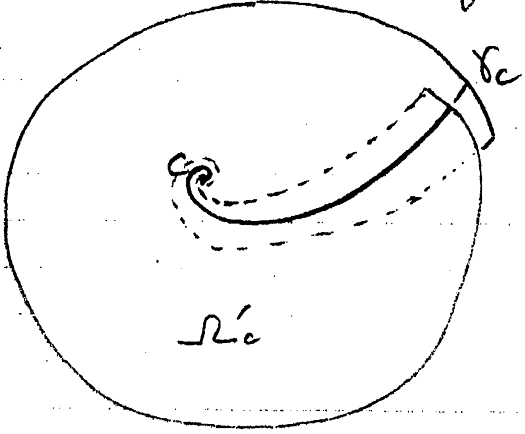
Let us begin with the following situation. Let $R(z) = z^2 + c$ be a quadratic polynomial on the Riemann sphere. We assume that the complex dynamical system defined by this quadratic polynomial is postcritically finite, i.e., the forward orbit $\{f^n(0) \mid n = 1, 2, \cdots\}$ of the critical point 0 of $R(z)$ is a finite set. For the sake of simplicity, we denote by F the Fatou set of $R(z)$ and by J the Julia set $\hat{\mathbb{C}} \setminus F$. In order to illustrate the situation, we consider especially the case $c = i$. Then the critical point is 0 and its forward orbit is $\{0, i, i-1, -i\}$ and $R(c)$ and $R^2(c)$ form a periodic cycle of period 2. $R(z)$ has two fixed points, which we denote by α and β as in the picture. These two fixed points are so-called the α-fixed point and the β-fixed point.



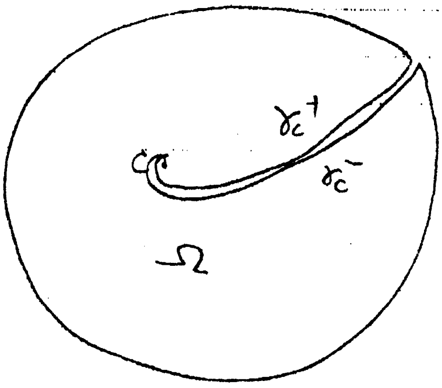
As $R(z)$ is postcritically finite, there exists an external ray, say γ_c, landing at the critical value c. We give an orientation to this curve as $\infty \to c$. Note that this external ray is spiraling near c.



Let $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$ be a domain in the complex plane. This domain Ω_c has smooth boundaries along the external ray γ_c. We consider an abstract "neighborhood" Ω_c' of Ω_c doubly sheeted near γ_c.



The domain Ω_c has two smooth curves on the boundary. We add two curves γ_c^+ and γ_c^- to this domain to each sides of the external ray γ_c, and we



denote this set by $\overline{\Omega}_c$. Ω_c' is an openset containing $\overline{\Omega}_c$. For domain Ω, we denote by $\mathcal{O}(\Omega)$ the set of holomorphic functions on Ω.

Let $f : \Omega_c \to \mathbb{C}$ be a holomorphic function on Ω_c which can be extended holomorphically to some "neighborhood" Ω_c' of $\overline{\Omega}_c$. Such a function f is said to be equivalent to $g : \Omega_c \to \mathbb{C}$ which is a holomorphic function on Ω_c and extendable to some "neighborhood" Ω_c'' of $\overline{\Omega}_c$ holomorphically, if there exists a "neighborhood" Ω_c''' of $\overline{\Omega}_c$ such that f and g coincides on Ω_c'''. This equivalence relation defines a concept of germ. Note that $f : \Omega_c \to \mathbb{C}$ itself gives a representative of its germ since analytic continuation is unique if it exists. We call such function f a (general) premicrofunction along γ_c. A (general) microfunction at c along γ_c is defined by an equivalence class of germs of (general) pre-microfunctions $f : \Omega_c \to \mathbb{C}$ at c modulo germs of holomorphic functions at c. More precisely, (general) pre-microfunctions $f : \Omega_c \to \mathbb{C}$ and $g : \Omega_c \to \mathbb{C}$ defines the same microfunction at c if there exists an open neighborhood U of c in the complex plane $\mathbb{C}$ and a holomorphic function $h : U \to \mathbb{C}$ such that $f(z) - g(z) = h(z)$ holds for $z \in U \cap \Omega_c$. The above definition of (general) microfunction is so general that the singularities of such functions at c are too much complicated. So, we restrict our singularities to "regular singularities" defined as follows.

<u>Definition</u> 1.1. Pre-microfunction $f : \Omega_c \to \mathbb{C}$ is said to have a regular singularity at c if there exist positive numbers ε and K such that inequality

$$|f(z)| < K \, |z - c|^{-1+\varepsilon}$$

holds near c.

<u>Definition</u> 1.2. Pre-microfunction $f : \Omega_c \to \mathbb{C}$ is said to have a regular singularity at ∞ if there exist positive numbers ε and K such that inequality

$$|f(z)| < K |z|^{-\varepsilon}$$

holds near the infinity.

We denote by $\mathcal{M}_c$ the set of pre-microfunctions along γ_c with regular singularities both at c and ∞. More precisely we denote $\mathcal{M}_{\gamma_c}$ instead of $\mathcal{M}_c$ when there are are more than one external rays landing at c. The space of equivalence classes of germs of pre-microfunctions with regular singularities at c modulo the space of germs of holomorphic functions $\mathcal{O}(c)$ at c, will be denoted by $\widetilde{\mathcal{M}}_c$.

Let $P(R)$ denote the postcritical set. For each point

$p \in \mathcal{P}(R)$, the space of pre-microfunctions along its external rays with regular singularities at both p and ∞ is defined in a similar manner and will be denoted by $\mathcal{M}_p$ for simplicity and by $\mathcal{M}_{\gamma_p}$ when it is necessary to indicate the external ray.

For $p \in J$ with multiple external rays, say $\gamma_1, \dots, \gamma_r$, landing at p, we define the space $\mathcal{M}_p$ by the direct sum

$$\mathcal{M}_p = \bigoplus_{k=1}^{r} \mathcal{M}_{\gamma_k}.$$

Where the sum is taken as a formal sum, since each component belongs to different spaces. However each element of $\mathcal{M}_p$ defines a function holomorphic in the intersection of the domains of definitions and the deomposition of a holomorphic function

$$f : \mathbb{C} \setminus \left(\left(\bigcup_{k=1}^{r} \gamma_k \right) \cup \{p\} \right) \longrightarrow \mathbb{C}$$

defined by an element of $\mathcal{M}_p$ into components f_k in $\mathcal{M}_{\gamma_k}$,

$$f_k : \mathbb{C} \setminus (\gamma_k \cup \{p\}) \longrightarrow \mathbb{C}$$

is unique since we are considering the pre-microfunctions with regular singularities at the infinity. We denote

$$\mathcal{M}_+ = \bigoplus_{p \in \mathcal{P}(R)} \mathcal{M}_p$$

$$\mathcal{M}_0 = \mathcal{M}_{\gamma_0^+} \oplus \mathcal{M}_{\gamma_0^-}$$

$$\mathcal{M}_- = \bigoplus_{k=1}^{\infty} \bigoplus_{p \in R^{-k}(0)} \mathcal{M}_p$$

and

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_0 \oplus \mathcal{M}_-.$$

Here, the origin 0 is the critical point of our quadratic map $R(z)$ and there are two external rays landing at 0, which are pre-images of the external ray γ_c. γ_0^+ and γ_0^- denotes the external angles $\frac{1}{12}$ and $\frac{7}{12}$ respectively. Note that γ_c is the external angle $\frac{1}{6}$, since it is mapped to period two cycle of external rays with angles $\frac{1}{3}$ and $\frac{2}{3}$. Here, the infinite direct sum is only in a formal sense.

§2 Difference operator and an exact sequence

Let $\mathcal{O}(\gamma_c)$ denote the space of holomorphic functions in a neighbourhood of the external ray γ_c. An element of $\mathcal{O}(\gamma_c)$ is represented by a continuous function $f : \gamma_c \to \mathbb{C}$ which can be extended to some neighborhood of γ_c holomorphically. The space of holomorphic functions along the external ray γ_c with regular singularities at both c and the infinity is defined by the following.

$$\mathcal{O}_0(\gamma_c) = \left\{ f \in \mathcal{O}(\gamma_c) \;\middle|\; \begin{array}{l} \exists \varepsilon > 0, \exists k > 0, \exists \text{ nbd of } \gamma_c \text{ s.t.} \\ |f(z)| < k|z-c|^{-1+\varepsilon} \quad \text{near } c \\ \text{and } |f(z)| < k|z|^{-\varepsilon} \quad \text{near } \infty \end{array} \right\}$$

Now, we define a difference operator along an external ray.

Definition 2.1 Difference operator $\Delta_c : \mathcal{M}_c \to \mathcal{O}_0(\gamma_c)$ is defined by the difference of boundary values along γ_c

$$\Delta_c \varphi(z) = \varphi(z) - \varphi((z-c)e^{-2\pi i} + c).$$

Here $z \in \gamma_c$ is consider as a point in the boundary of Ω_c of the clockwise side and $(z-c)e^{-2\pi i} + c$ represents the same point but considered as a point in the boundary of Ω_c of the counterclockwise side.

For each $p \in J$ and its external ray γ_p, difference operator $\Delta_p : \mathcal{M}_p \to \mathcal{O}_0(\gamma_p)$ is defined in a similar way. We denote Δ_{γ_p} instead of Δ_p if there are more than two external rays landing at p and we need to indicate it.

Remark Difference operator can be defined for functions holomorphic along γ_c in a doubly sheeted domains. The domain of definition of such a holomorphic function need not be connected.

Let us fix a double sheeted neighborhood Ω_c' of our domain $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$, and let S_c denote the neighborhood of γ_c where Ω_c' is double sheeted.

Theorem 2.2 The following sequence is exact.

$$0 \to \mathcal{O}(\mathbb{C}\setminus\{c\}) \hookrightarrow \mathcal{O}(\Omega'_c) \xrightarrow{\Delta_c} \mathcal{O}(S_c) \longrightarrow 0.$$

Proof We gave an orientation to the external ray γ_c defining an order to the points in γ_c so that $\infty < p < c$. Take points $r_j, s_j \in \gamma_c$ for $j \in \mathbb{Z}$ ordered along γ_c as

$$\infty < \cdots < r_j < s_{j-1} < r_{j+1} < s_j < \cdots < c$$

and

$$\lim_{k\to-\infty} s_k = \lim_{k\to-\infty} r_k = \infty,$$

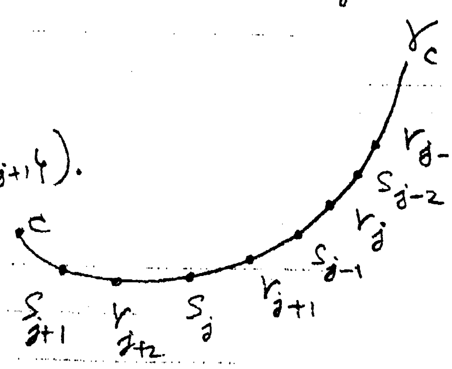
$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} r_k = c.$$

Then open arcs $\widehat{r_j s_j}$ $(j \in \mathbb{Z})$ form an open covering of the external ray γ_c. The space $\mathcal{O}(\mathbb{C}\setminus\{c\})$ of holomorphic functions on $\mathbb{C}\setminus\{c\}$ can be injectively embedded in the space of holomorphic functions on Ω'_c and the values of such a function on the two sheets of Ω'_c coincide on the overlapped sector S_c, hence the difference of these values vanishes. So, we need only to prove the ontoness of the difference operator Δ_c. For $\varphi \in \mathcal{O}(S_c)$, we want to construct a holomorphic function in $\mathcal{O}(\Omega'_c)$. Note that such a function is not unique since the kernel of Δ_c contains $\mathcal{O}(\mathbb{C}\setminus\{c\})$.

Let

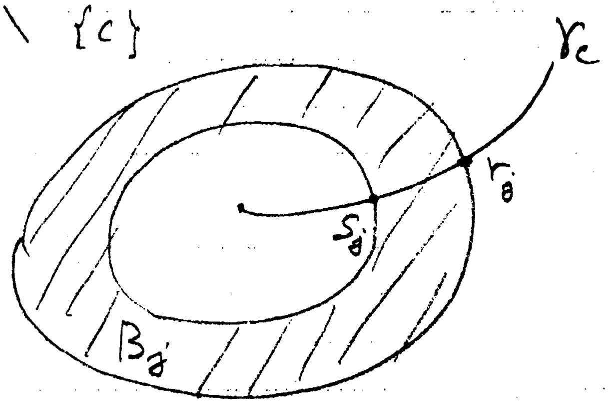
$$F_j(z) = \frac{1}{2\pi i}\int_{r_{j-1}}^{s_{j+1}} \frac{\varphi(\tau)}{\tau - z}\, d\tau$$

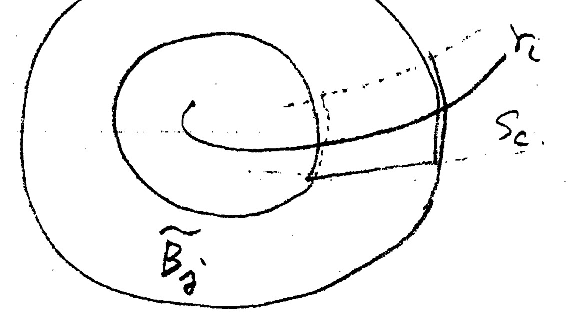
for $j \in \mathbb{Z}$. Such integration is called a Cousin's integral along the arc $\widehat{r_{j-1} s_{j+1}}$. Note that this arc includes the arc $\widehat{r_j s_j}$ in its interior. The function $F_j(z)$ is holomorphic in $\mathbb{C}\setminus(\widehat{r_{j-1} s_{j+1}} \cup \{r_{j-1}\} \cup \{s_{j+1}\})$. By deforming the path of integration of the Cousin's integral we see that $F_j(z)$ can be holomorphically extended beyond the arc from both sides into the other sides, except



at r_{j-1} and s_{j+1}. Next let us take a family of annuli in $\mathbb{C} \setminus \{c\}$ separating c and ∞ with smooth boundaries as follows. We take anulus B_j for each $j \in \mathbb{Z}$ so that the intersection of B_j with the external ray γ_c is the arc $\overline{r_j s_j}$, and r_j, s_j belong to the outer and inner boundary of B_j respectively. Furthermore, we for $j, k \in \mathbb{Z}$, $B_j \cap B_k$ is empty if $|k - j| > 1$ hold, and for each $j \in \mathbb{Z}$, $B_j \cap B_{j+1}$ is an an annulus. We impose that

$$\bigcup_{j \in \mathbb{Z}} B_j = \mathbb{C} \setminus \{c\}$$





For each j, we denote by $\widetilde{B}_j$ a covering of B_j such that $\widetilde{B}_j$ covers twice on the sector $S_c \cap B_j$. Our function $F_j(z)$ defined by Cousin's integration can be extended holomorphically to $\widetilde{B}_j$. It is further extendable to a wider domain $\widetilde{B}_{j-1} \cup \widetilde{B}_j \cup \widetilde{B}_{j+1}$. Hence $F_j(z)$ is Bonded in $\widetilde{B}_j$. As is easily verified by considering the integration, we have

$$F_j(z) - F_j\left((z-c)e^{-2\pi i} + c\right) = \varphi(z)$$

for $z \in S_c \cap B_j$.

For $j, k \in \mathbb{Z}$ with $B_j \cap B_j \neq \phi$, define a holomorphic function

$$H_{jk} : B_j \cap B_k \to \mathbb{C}$$

by

$$H_{jk}(z) = F_j(z) - F_k(z).$$

$F_j(z)$ and $F_k(z)$ are holomorphic on $\widetilde{B}_j \cap \widetilde{B}_j$. But, as we have

$$F_j((z-c)e^{-2\pi i} + c) - F_k(z-c)e^{-2\pi i} + c)$$
$$= F_j(z) - F_k(z)$$

along γ_c, $H_{jk}((z-c)e^{-2\pi i}+c) = H_{jk}(z)$ holds on $S_c \cap B_j \cap B_k$, so that $H_{jk}(z)$ is well defined and holomorphic on the annulus $B_j \cap B_k$. This family of holomorphic functions $\{H_{jk}\}$ forms a "Cousin data", i.e. for $i, j, k \in \mathbb{Z}$,

$$H_{ij} + H_{jk} + H_{ki} = 0 \quad \text{on } B_i \cap B_j \cap B_k.$$

As we assumed $B_j \cap B_k = \phi$ if $|j-k|>1$, the above fact is easily verified.

For each $j \in \mathbb{Z}$, take a loop l_j in $B_j \cap B_{j+1}$, making a clockwise turn once and define $h_j(z)$ and $k_j(z)$ by

$$h_j(z) = \frac{1}{2\pi i}\int_{l_j} \frac{H_{i,i+1}(\tau)}{\tau - z}\, d\tau$$

defined and holomorphic in $\bigcup_{k=-\infty}^{j} B_k \cup \{\infty\}$ (outside of the annulus), and

$$k_{j+1}(z) = \frac{1}{2\pi i}\int_{\gamma_j} \frac{H_{i,i+1}(\tau)}{\tau - z}\, d\tau$$

define an holomorphic in $\bigcup_{k=j+1}^{\infty} B_k \cup \{c\}$ (inside of the annulus).

By deforming the integration path we see that they are well defined and we have

$$H_{j,j+1}(z) = h_j(z) - k_{j+1}(z) \quad (z \in B_j \cap B_{j+1})$$

By Runge's theorem, $k_{j+1} : \bigcup_{k=j+1}^{\infty} B_k \cup \{c\} \to \mathbb{C}$ can be appoximated by polynomials in the sense of uniform convergence on compact sets, and $h_j : \bigcup_{k=-\infty}^{j} B_k \cup \{\infty\}$ can be approximated by rational functions with poles only at c.

For each $j \geq 0$, find a rational function $g_j : \bar{\mathbb{C}} \setminus \{c\} \to \mathbb{C}$ such that

$$|g_j(z) - h_j(z)| < \frac{1}{2^j} \quad \text{for } z \in \bigcup_{i=-\infty}^{j} B_i \cup \{\infty\}.$$

And for each $j < 0$, find a polynomial $g_j : \mathbb{C} \to \mathbb{C}$ such that

$$|g_j(z) - k_{j+1}(z)| < \frac{1}{2^{|j|}} \quad \text{for } z \in \bigcup_{i=j+1}^{\infty} B_i \cup \{c\}.$$

Note that these functions g_j are all holomorphic in $\mathbb{C} \setminus \{c\}$.

Let for $j \geq 0$, $\tilde{h}_j = h_j - g_j : \bigcup_{i=-\infty}^{j} B_i \cup \{\infty\} \to \mathbb{C}$
and
$$\tilde{k}_{j+1} = k_{j+1} - g_j : \bigcup_{i=j+1}^{\infty} B_i \cup \{c\} \to \mathbb{C},$$
for $j < 0$. Then we have
$$|\tilde{h}_j| < \frac{1}{2^j} \quad (j \geq 0)$$
and
$$|\tilde{k}_{j+1}| < \frac{1}{2^{|j|}} \quad (j < 0).$$
We still have
$$H_{j,j+1}(z) = \tilde{h}_j(z) - \tilde{k}_{j+1}(z) \quad \text{for } z \in B_j \cap B_{j+1}.$$
Now, we set
$$H_j(z) = -\sum_{i=-\infty}^{j} \tilde{k}_i(z) - \sum_{i=j}^{\infty} \tilde{h}_i(z).$$
For $i \leq j$, $\tilde{k}_i(z)$ is holomorphic in $\bigcup_{l=i}^{\infty} B_l \cup \{c\}$, hence they are all holomorphic in the smallest disk $\bigcup_{l=j}^{\infty} B_l \cup \{c\}$ and that we have the estimate of the supremum of the functions, the sum of $\tilde{k}_i$'s is uniformly convergent on B_j. Similarly, the sum of $\tilde{h}_i$'s converge uniformly convergent on B_j, too. Hence $H_j(z)$ is holomorphic in B_j.

In the overlapping annulus $B_j \cap B_{j+1}$, we have
$$\begin{aligned} H_{j+1} - H_j &= -\sum_{i=-\infty}^{j+1} \tilde{k}_i - \sum_{i=j+1}^{\infty} \tilde{h}_i + \sum_{i=-\infty}^{j} \tilde{k}_i + \sum_{i=j}^{\infty} \tilde{h}_i \\ &= \tilde{h}_j - \tilde{k}_{j+1} = H_{j,j+1}. \end{aligned}$$
Finally, in $\widehat{B}_j$, let $G_j(z) = H_j(z) + F_j(z)$. These functions $\{G_j\}$ on $\widehat{B}_j$ defines a holomorphic function
$$G : \Omega_c' \to \mathbb{C}$$
define on the overlapped neighborhood Ω_c' of Ω_c. We can verify that these functions coincide and G is well defined by an immediate calculations as follows.
In $\widehat{B}_j \cap \widehat{B}_{j+1}$,
$$\begin{aligned} G_{j+1}(z) &= H_{j+1}(z) + F_{j+1}(z) = H_j(z) + H_{j,j+1}(z) + F_{j+1}(z) \\ &= H_j(z) + F_j(z) - F_{j+1}(z) + F_{j+1}(z) = H_j(z) + F_j(z) = G_j(z) \end{aligned}$$

Thus, we conclude that $G \in \mathcal{O}(\Omega_c')$ and

$$\Delta_c G = \varphi$$

holds. This completes the proof the our Theorem 2.2.

We remark that such a function G satisfying $\Delta_c G = \varphi$ is not unique since $\operatorname{Ker} \Delta_c = \mathcal{O}(\mathbb{C} \setminus \{c\})$.

§3. Cousin's integral operator and decomposition of pro-microfunctions.

In the previous section, we discussed the surjectivity of the difference operator Δ_c. In this section, we restrict the space of (general) pre-microfunctions to the space of pre-microfunctions with regular singularities, and consider an inverse operator of Δ_c, which we call a Cousin's integral operator.

Definition 3.1 $\mathcal{I}_c : \mathcal{O}_0(\gamma_c) \longrightarrow \mathcal{M}_c$ is defined by

$$\mathcal{I}_c[\varphi](z) = \frac{1}{2\pi i}\int_{\gamma_c} \frac{\varphi(\tau)}{\tau - z}\, d\tau$$

for $\varphi \in \mathcal{O}_0(\gamma_c)$.

Here, we use notation $\mathcal{I}_c[\varphi]$ as $\mathcal{I}_c : \mathcal{O}_0(\gamma_c) \longrightarrow \mathcal{M}_c$ is an operator and we want to emphasise it, i.e. the arguement of the operator is a function and not its value.

Definition 3.2 Let f be bi-valued function defined in a neighborhood of γ_c, both of the two branches are holomorphic and the differece $\Delta_c f$ of f has regular singularities at c and at ∞, i.e. $\Delta_c f \in \mathcal{O}_0(\gamma_c)$. The $\mathcal{M}_c$ component of f is defined as

$$[f]_c = [f]_{\gamma_c} = \mathcal{I}_c[\Delta_c f].$$

This mapping $[\ ]_c$ is a projection map onto $\mathcal{M}_c$.

We have the following identities.

<u>Theorem</u> 3.3
$$\mathcal{I}_c \circ \Delta_c = id \quad \text{on } \mathcal{M}_c,$$
$$\Delta_c \circ \mathcal{I}_c = id \quad \text{on } \mathcal{O}_0(\gamma_c).$$

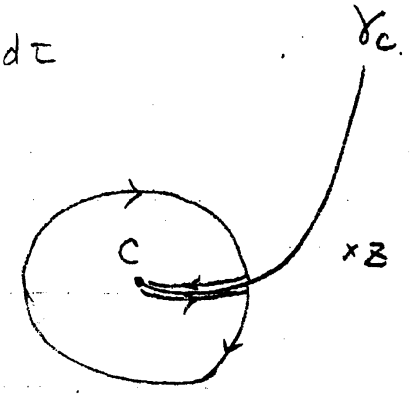
<u>Proof</u> These identities are easily verified.

For each point $p \in J$ (and an external ray γ_p landing at p), projection $[\ ]_p$ is similarly defined.

Let $\mathcal{O}_0(\Omega_c)$ denote the space of holomorphic functions $f : \Omega_c \to \mathbb{C}$ such that f can be extendable holomorphically to some double sheeted neighborhood Ω_c' and satisfies $\Delta_c f \in \mathcal{O}_0(\gamma_c)$. Function $f \in \mathcal{O}_0(\Omega_c)$ is holomorphic in $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$ and has singularities at c and at the infinity together with it's difference along γ_c.

Let $\mathcal{H}_c$ denote the space of hyperfunctions supported at c, i.e. $\varphi \in \mathcal{H}_c$ if and only if φ is holomorphic in $(\mathbb{C} \cup \{\infty\}) \setminus \{c\}$. The space of entire functions is denoted by $\mathcal{O}(\mathbb{C})$. Let us define the operators that extract singularities of f.

<u>Definition</u> 3.4. Operator $\Gamma_c : \mathcal{O}_0(\Omega_c) \to \mathcal{H}_c$ is defined by
$$\Gamma_c[f](z) = \frac{1}{2\pi i} \int_{|\tau - c| = \varepsilon} \frac{f(\tau)}{\tau - z}\, d\tau$$
$$- \frac{1}{2\pi i} \int_{r_\varepsilon}^{c} \frac{(\Delta_c f)(\tau)}{\tau - z}\, d\tau$$



where $z \in \mathbb{C} \setminus \{c\}$, $\varepsilon > 0$ is chosen sufficiently small so that the ε-ball around c does not contain z, and $r_\varepsilon \in \gamma_c$ is the intersection point of γ_c and the circle $|z - c| = \varepsilon$. The orientation of the path of integration along the circle is the counterclockwise with respect to z.

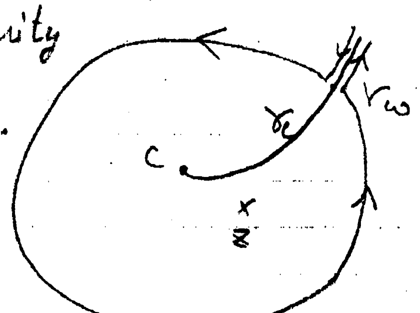
As $\Delta_c f$ has a regular singularity at c, this defines a holomorphic function on $(\mathbb{C}\cup\{\infty\})\setminus\{c\}$. That is, $\Gamma_c[f] \in \mathcal{H}_c$.

<u>Definition</u> 3.5. Operator $\Gamma_\infty : \mathcal{O}_0(\Omega_c) \longrightarrow \mathcal{O}(\mathbb{C})$ is defined by

$$\Gamma_\infty[f](z) = \frac{1}{2\pi i}\int_{|\tau-c|=w} \frac{f(\tau)}{\tau - z}\,d\tau + \frac{1}{2\pi i}\int_{r_w}^{\infty} \frac{(\Delta_c f)(\tau)}{\tau - z}\,d\tau$$

when $w>0$ is taken sufficiently large for each z, so that the circle of integration path surrounds z, and $r_w \in \gamma_c$ is the intersection point of γ_c and the big circle. The orientation is taken as the counterclockwise with respect to z.

As $\Delta_c f$ has a regular singularity at ∞, this defines an entire function. Hence $\Gamma_\infty[f] \in \mathcal{O}(\mathbb{C})$.



Just for the sake of consistence of notation we define

$$\Gamma_M : \mathcal{O}_0(\Omega_c) \longrightarrow \mathcal{M}_c$$

by $\Gamma_M[f] = [f]_c$. We have the following decomposition.

<u>Theorem</u> 3.6. $\mathcal{O}_0(\Omega_c) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C})$
and $\Gamma_c : \mathcal{O}_0(\Omega_c) \rightarrow \mathcal{H}_c$, $\Gamma_M : \mathcal{O}_0(\Omega_c) \rightarrow \mathcal{M}_c$
$\Gamma_\infty : \mathcal{O}_0(\Omega_c) \rightarrow \mathcal{O}(\mathbb{C})$
gives the projections to components.

<u>Proof</u> Clearly, the kernel of the defference operator Δ_c is $\mathcal{O}(\mathbb{C}\setminus\{c\})$ and $\mathcal{O}(\mathbb{C}\setminus\{c\}) = \mathcal{H}_c \oplus \mathcal{O}(\mathbb{C})$.

Note that these operators can be difined if f is defined and holomorphic in a double covered neighborhood of γ_c. In this case, $\Gamma_c + \Gamma_M + \Gamma_\infty$ defines a projection to $\mathcal{O}_0(\Omega_c) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C})$ if $\Delta_c f \in \mathcal{O}_0(\gamma_c)$.

§4 Space of pre-microfunctions and a transfer operator

Let us go back to our complex dynamical system $R(z)$. Let J denote the Julia set of $R(z)$ and let F denote the Fatou set of $R(z)$. We suppose $R(z)$ is postcritically finite, especially the case of $c=i$. We denote by $\mathcal{O}(J)$ the space of germs of continuous functions $f: J \to \mathbb{C}$ which are holomorphic in some neighborhood of J. The space of holomorphic functions $f: F \to \mathbb{C}$ of the Fatou set satisfying $f(\infty)=0$ will be denoted by $\mathcal{O}_0(F)$. The postcritical set of $R(z)$ is denoted by $P(R)$. The space of pre-microfunctions at $P(R)$ is defined by

$$\mathcal{M}_{P(R)} = \bigoplus_{p \in P(R)} \mathcal{M}_p.$$

and

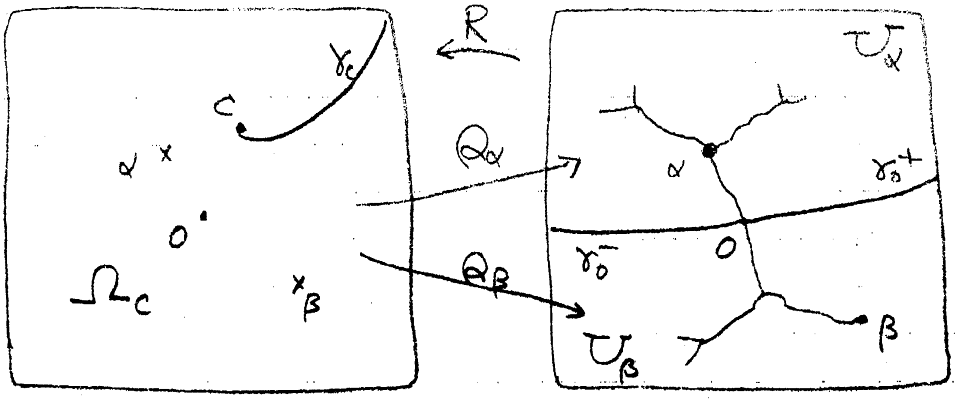
$$\mathcal{M} = \mathcal{M}_{P(R)} \oplus \bigoplus_{k=0}^{\infty} \bigoplus_{p \in R^{-k}(0)} \mathcal{M}_p$$

denotes the space of formal sup of premicrofunctions at the grand obit of the critical point 0.

Let $f \in \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F)$ and $p \in P(R)$ with γ_p its external ray. Then the $\mathcal{M}_p$-component $[f]_p$ of f is given by a projection

$$[f]_p(z) = \frac{1}{2\pi i} \int_{\gamma_p} \frac{(\Delta_p f)(z)}{\tau - z} d\tau.$$

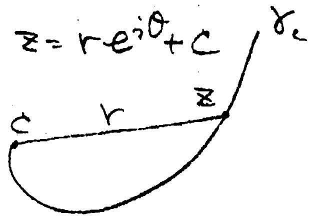
Let us consider the most simple postcritically finite case (except $c=-2$ case) of $R(z) = z^2 + i$. Fixed points of R are denoted by α and β. The preimage of the external ray γ_c consists of two external rays, say γ_0^+ and γ_0^-, of the critical point 0, with external angles $\frac{1}{12}$ and $\frac{7}{12}$ respectively. These external rays are oriented as $\infty \to 0$. Let U_α denote the upper connected component of $\mathbb{C} \setminus (\gamma_0^+ \cup \gamma_0^-)$ which contains the critical value $c=i$. The α-fixed point belongs to this domain. We denote the other connected point by U_β. It contains the β-fixed point. The quadratic map R is of degree two. The critical value c is a branch point. We denote the two branches of R^{-1} by Q_α and Q_β defined in $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$.



$$Q_\alpha : \mathbb{C} \setminus (\gamma_c \cup \{c\}) \longrightarrow U_\alpha$$
$$Q_\beta : \mathbb{C} \setminus (\gamma_c \cup \{c\}) \longrightarrow U_\beta$$

with $Q_\alpha(z) = -\sqrt{z-c}$, $Q_\beta(z) = \sqrt{z-c}$, where the branch of the square root is chosen by assigning $Q_\beta(c+1)=1$. If we regard Q_α and Q_β as holomorphic functions on Ω_c, we can naturally consider holomorphic functions $(Q_\alpha(z))^s$ and $(Q_\beta(z))^s$ for $0 < s < 2$. They can be extended to a double sheeted neighborhood Ω_c' holomorphically. We define a holomorphic function

$$\Psi_s(z) = \frac{1}{(2Q_\beta(z))^s}$$



defined in Ω_c'.

For $z = c + re^{i\theta} \in \gamma_c$, we have

$$(\Delta_c \Psi_s)(z) = \Psi_s(z) - \Psi_s((z-c)e^{-2\pi i} + c)$$
$$= \frac{1 - e^{s\pi i}}{(2\sqrt{r}\, e^{\frac{\theta}{2} i})^s}$$

Hence
$$|\Delta_c \Psi_s| = \text{const. } r^{-\frac{s}{2}}$$
which implies $\Delta_c \Psi_s$ has regular singularities at c and ∞ if $0 < s < 2$. Therefore $\Psi_s \in \mathcal{M}_c$.

Now, take a function $f \in \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F)$. Here, we abuse the formal sum of function in different spaces and the sum as a functions defined in the common domain of definition. So, f is defined and holomorphic in $F \setminus (\bigcup_{p \in P(R)} \gamma_p)$. For an external ray γ_p, we denote by

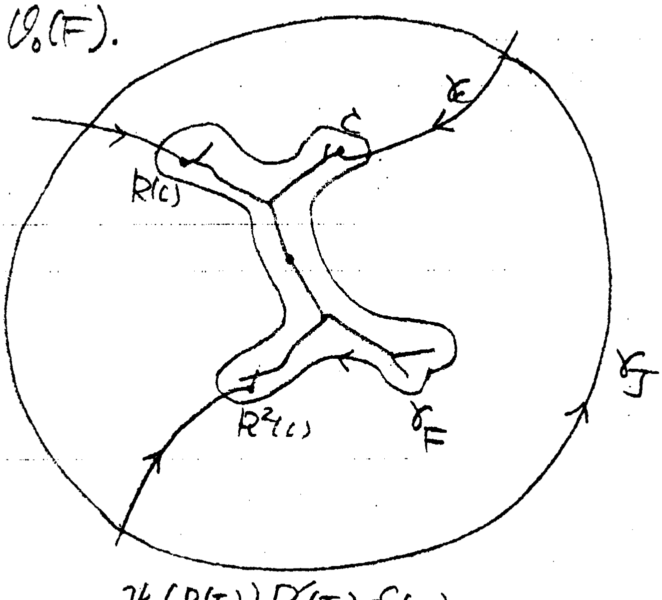
$\tilde{\gamma}_p$ the path of integration coming from ∞ to p along γ_p taking the clockwise side value of the integrand function and going back from p to ∞ along γ_p taking the counterclockwise side value of the integrand function. That is,

$$\int_{\tilde{\gamma}_p} f(z)dz = \int_{\gamma_p} f(z)\,dz - \int_{\gamma_p} f((z-p)e^{2\pi i}+p)\,dz$$

$$= \int_{\gamma_p} (\Delta_p f)(z)\,dz.$$

By γ_F we represent an integration path along the boundary of the Fatou set, passing near the Julia set and taking values of the function on the Fatou set. And finally by γ_J we represent an integration path turning around the Julia set in the counterclockwise direction.

Let $\mathcal{H}_+ = \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F)$.



<u>Definition</u> 4.1 Transfer operator $\mathcal{L}_s : \mathcal{H}_+ \to \mathcal{H}_+$ is defined for $0<s<2$ and for $f \in \mathcal{H}_+$ by

$$(\mathcal{L}_s f)(z) = \sum_{y \in R^{-1}(z)} \Psi_s(R(y)) f(y).$$

We can rewrite the transfer operator in an integral operator form as

$$(\mathcal{L}_s f)(z) = \frac{1}{2\pi i} \int_{\gamma_F + \tilde{\gamma}_{P(R)} + \gamma_J} \frac{\Psi_s(R(\tau))\, R'(\tau)\, f(\tau)}{R(\tau) - z}\, d\tau$$

and as

$$(\mathcal{L}_s f)(z) = \Psi_s(z)\left(f\circ Q_\alpha(z) + f\circ Q_\beta(z)\right).$$

<u>Definition</u> 4.2 Push forward operator R_* is defined by

$$R_* f = f\circ Q_\alpha + f\circ Q_\beta.$$

For point $\eta \in \mathbb{C}$, we denote by $\chi_\eta(z) = \frac{1}{z-\eta}$ the unit pole at η. If $\eta \in J$ then $\chi_\eta \in \mathcal{O}_0(F)$. If $\eta \in \gamma_p$ then $\chi_\eta \in \mathcal{O}(\Omega_p)$. Note that $\chi_z(\eta) = -\chi_\eta(z)$.

<u>Proposition</u> 4.3 For $\eta \in J \cup \bigcup_{p \in P(R)} \gamma_p$,

$$\mathcal{L}_s \chi_\eta = R'(\eta)\,\psi_s(R(\eta))\chi_{R(\eta)} + R'(\eta)[\psi_s \cdot \chi_{R(\eta)}]_C .$$

<u>Proof</u> By a direct computation, we have

$$(\mathcal{L}_s \chi_\eta)(z) = \psi_s(z)\left(\chi_\eta \circ Q_\alpha(z) + \chi_\eta \circ Q_\beta(z)\right) = \psi_s(z) \sum_{y \in R^{-1}(z)} \left(-\chi_y(\eta)\right)$$

$$= \psi_s(z) \sum_{y \in R^{-1}(z)} \frac{1}{y-\eta}.$$

By the residue formula, we have

$$0 = \frac{1}{2\pi i}\int_{\gamma_J} \frac{R'(y)}{(z-R(y))(y-\eta)}\,dy = \sum_{y \in R^{-1}(z)} \frac{1}{y-\eta} + \frac{R'(\eta)}{z-R(\eta)}.$$

Hence $(\mathcal{L}_s \chi_\eta)(z) = \psi_s(z)\,\frac{R'(\eta)}{z-R(\eta)} = \psi_s(z) R'(\eta)\chi_{R(\eta)}(z)$

$$= \psi_s(R(\eta))\,R'(\eta)\chi_{R(\eta)}(z) + [R'(\eta)\cdot \psi_s \cdot \chi_{R(\eta)}]_C .$$

Here $[R'(\eta)\psi_s \cdot \chi_{R(\eta)}]_C(z) = (\psi_s(z) - \psi_s(R(\eta)))\,R'(\eta)\,\chi_{R(\eta)}(z)$ is holomorphic near $R(\eta)$ so that it belongs to $\mathcal{M}_C$. The first term $\psi_s(R(\eta))\,R'(\eta)\,\chi_{R(\eta)}(z)$ is a multiple of unit pole at $R(\eta)$.

Unit poles $\{\chi_\eta\}_{\eta \in J}$ form a basis of function space $\mathcal{O}(F)$ and the family of unit poles $\{\chi_\eta\}_{\eta \in \gamma_p}$ form a basis of space of pre-microfunctions $\mathcal{M}_p$. This splitting of the transfer operator $\mathcal{L}_s$ gives a decomposition of the operator into components of the direct sum decomposition of function spaces.

<u>Theorem</u> 4.4 For $p \in J$ and $g \in \mathcal{M}_p$ we have the following decomposition

$$\mathcal{L}_s g = [g \circ Q_p \cdot \psi_s]_{R(p)} + [\psi_s \cdot [g \circ Q_p]_{R(p)}]_C$$

where $Q_p = Q_\alpha$ or Q_β according to $p \in U_\alpha$ or U_β.

Proof. As $g \in \mathcal{M}_p$, $g(z)$ can be expressed in an integration of Cauchy type for $z \in \Omega_p = \mathbb{C} \setminus (\gamma_p \cup \{p\})$ as

$$g(z) = \frac{1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(t)\frac{dt}{t-z} = \frac{1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(t)(-\chi_t(z))dt.$$

Hence, we have

$$\begin{aligned}
(\mathcal{L}_s g)(z) &= \frac{-1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(\eta)(\mathcal{L}_s\chi_\eta)(z)d\eta \\
&= \frac{-1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(\eta)\left(\psi_s(R(\eta))\cdot R'(\eta)\chi_{R(\eta)} + [\psi_s\cdot R'(\eta)\cdot\chi_{R(\eta)}]_c\right)d\eta \\
&= \frac{-1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(\eta)\cdot\psi_s(R(\eta))\cdot R'(\eta)\cdot\chi_{R(\eta)}\,d\eta - \frac{1}{2\pi i}\int_{\gamma_p}(\Delta_p g)(\eta)[\psi_s\cdot R'(\eta)\cdot\chi_{R(\eta)}]_c\,d\eta \\
&= \frac{-1}{2\pi i}\int_{\gamma_{R(p)}} (\Delta_{R(p)}(g\circ Q_p))(\xi)\cdot\psi_s(\xi)\chi_\xi\,d\xi + \left[\psi_s\cdot\frac{-1}{2\pi i}\int_{\gamma_p}(\Delta_p g)(\eta)\cdot R'(\eta)\chi_{R(\eta)}d\eta\right]_c \\
&= [(g\circ Q_p)\cdot\psi_s]_{R(p)} + [\psi_s\cdot[g\circ Q_p]_{R(p)}]_c .
\end{aligned}$$

Note that in the above calculations, $\eta \in \gamma_p$ is a variable along γ_p and we changed variables by $\xi = R(\eta)$ and $d\xi = R'(\eta)d\eta$.

This theorem shows that the transfer operator $\mathcal{L}_s$ sends $\mathcal{M}_p$ into $\mathcal{M}_{R(p)} \oplus \mathcal{M}_C$ for $p \in J$. In the case of postcritically finite complex dynamical system case, the space of pre-microfunctions for postcritical set is invariant for $\mathcal{L}_s$, i.e.

Proposition $\mathcal{L}_s(\mathcal{M}_+) \subset \mathcal{M}_+$.

§5. Decomposition of the transfer operator

Our space of pre-microfunctions $\mathcal{H}_+$ has a direct sum decomposition

$$\mathcal{H}_+ = \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_+ \oplus \mathcal{O}_0(F)$$

and the transfer operator $\mathcal{L}_s$ maps this space into itself. We denote the components of $\mathcal{L}_s$ in a matrix form as

$$\mathcal{L}_s = \begin{pmatrix} \mathcal{L}_{JJ} & \mathcal{L}_{JM} & \mathcal{L}_{JF} \\ \mathcal{L}_{MJ} & \mathcal{L}_{MM} & \mathcal{L}_{MF} \\ \mathcal{L}_{FJ} & \mathcal{L}_{FM} & \mathcal{L}_{FF} \end{pmatrix}.$$

For $f_J \in \mathcal{O}(\mathbb{C})$ we have the following proposition.

Proposition 5.1 $\mathcal{L}_s f_J = \Psi_s \cdot R_* f_J = \mathcal{L}_{JJ} f_J + \mathcal{L}_{MJ} f_J + \mathcal{L}_{FJ} f_J$
with

$$\mathcal{L}_{JJ} f_J = \Psi_s \cdot R_* f_J - [\Psi_s \cdot R_* f_J]_c,$$
$$\mathcal{L}_{MJ} f_J = [\Psi_s \cdot R_* f_J]_c,$$
$$\mathcal{L}_{FJ} f_J = 0.$$

Proof As $0 < s < 2$, we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_\rho} \frac{\Psi_s(R(\tau)) \cdot R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau = 0,$$

where $z \neq c$ and the integration path C_ρ is the circle of radius ε around the critical point 0, since the singularity at the origin is of order $1-s$. In the next calculation, integration paths are as explained in the previous section. We have

$$(\mathcal{L}_{JJ} f_J)(z) = \frac{1}{2\pi i} \int_{\gamma_J} \frac{\Psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau$$

$$= \frac{1}{2\pi i} \left\{ \int_{\gamma_J} + \int_{\tilde{\gamma}_0^+ + \tilde{\gamma}_0^-} - \int_{\tilde{\gamma}_0^+ + \tilde{\gamma}_0^-} + \int_{C_\rho} - \int_{C_\rho} \right\} \frac{\Psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau$$

$$= \Psi_s(z) \sum_{y \in R^{-1}(z)} f_J(y) - \frac{1}{2\pi i} \int_{\tilde{\gamma}_0^+ + \tilde{\gamma}_0^-} \frac{\Psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau$$

$$= \Psi_s(z) \sum_{y \in R^{-1}(z)} f_J(y) - \frac{1}{2\pi i} \int_{\gamma_0^+ + \gamma_0^-} \frac{(\Delta_c \Psi_s)(R(\tau)) f_J(\tau) \cdot R'(\tau) d\tau}{R(\tau) - z}$$

$$= \Psi_s(z) (R_* f_J)(z) - \frac{1}{2\pi i} \int_{\gamma_c} \frac{(\Delta_c \Psi_s)(\sigma)(f_J \circ Q^+(\sigma) + f_J \circ Q^-(\sigma))}{\sigma - c} d\sigma$$

Here we made a change of variables by $\sigma = R(\tau)$ and $d\sigma = R'(\tau) d\tau$. $Q^+ : \gamma_c \to \gamma_0^+$ and $Q^- : \gamma_c \to \gamma_0^-$ denotes the inverse branches of R along γ_c. $R_* f_J = f_J \circ Q^+ + f_J \circ Q^-$ holds along γ_c and $R_* f_J = f_J \circ Q_\alpha + f_J \circ Q_\beta$ elsewhere. We continue the calculation.

$$(\mathcal{L}_{JJ} f_J)(z) = \Psi_s(z) \cdot (R_* f_J)(z) - I_c [(\Delta_c \Psi_s) \cdot R_* f_J]$$

$$= \Psi_s(z) \cdot R_* f_J(z) - [\Psi_s \cdot R_* f_J]_c.$$

This completes the first line. Note that $\mathcal{L}_s f_J = \Psi_s \cdot R_* f_J$ has singularities along γ_c only. Next, compute the component $\mathcal{L}_{MJ} f_J$ as follows.

$$(\mathcal{L}_{MJ} f_J)(z) = \frac{1}{2\pi i}\int_{\gamma_M} \frac{\Psi_s(R(\tau))\, R'(\tau)\, f_J(\tau)}{R(\tau) - z}\, d\tau$$

$$= [\Psi_s \cdot R_* f_J]_c$$

and

$$(\mathcal{L}_{FJ} f_J)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\Psi_s(R(\tau))R'(\tau) f_J(\tau)}{R(\tau) - z}\, d\tau$$

$$= \frac{1}{2\pi i}\int_{|\sigma - c| = \rho} \frac{\Psi_s(\sigma)(R_* f_J)(\sigma)}{\sigma - z}\, d\sigma$$

$$= 0$$

since $|\mathcal{L}_{FJ} f_J| \leq \frac{2\pi\rho}{2\pi}\, \frac{|2\rho|^{1-s}\, |f_J(0) + f_J'(0)\rho|}{|z-c| - \rho} \longrightarrow 0$ (as $\rho \to 0$)

For the second column of the operator matrix, we have the following.

<u>Proposition</u> 5.2. For $f_M \in \mathcal{M}_p \subset \mathcal{M}_+$ $(p \neq 0)$,

$$\mathcal{L}_{JM} f_M = 0$$
$$\mathcal{L}_{MM} f_M = [\Psi_s \cdot R_* f_M]_c + [\Psi_s \cdot f_M \circ Q_p]_{R(p)}$$
$$\mathcal{L}_{FM} f_M = 0$$

<u>Remark</u> $\mathcal{L}_s f_M \in \mathcal{M}_c \oplus \mathcal{M}_{R(p)}$ if $f_M \in \mathcal{M}_p$

<u>Proof</u> As $f_M \in \mathcal{M}_p$, there exists a positive number ε and a positive constant K such that

$$|f_M(z)| < K|z|^{-\varepsilon}$$

holds near the infinity. Hence we have

$$|(\mathcal{L}_{JM} f_M)(z)| \leq \frac{1}{2\pi}\int_{\gamma_J} \frac{|\Psi_s(R(\tau))\, R'(\tau)\, f_M(\tau)|}{|R(\tau) - z|}\, d\tau$$

$$\leq \frac{2\pi|\tau|}{2\pi}\, |2\tau|^{1-s}\, \frac{1}{|\tau|^2 \left|1 + \frac{c}{\tau^2} - \frac{z}{\tau^2}\right|} K |\tau|^{-\varepsilon}$$

$$\leq \text{Const.}\, |\tau|^{-s-\varepsilon} \longrightarrow 0 \quad (|\tau| \to \infty)$$

and $\mathcal{L}_{JM} f_M = 0$.

Next, we show that $\mathcal{L}_{FM} = 0$ in the following.

$$(\mathcal{L}_{FM} f_M)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\psi_s(R(\tau))\,R'(\tau)\,f_M(\tau)}{R(\tau)-z}\,d\tau$$

$$= \frac{1}{2\pi i}\left\{\int_{|\tau-p|=\rho} + \int_{|\tau|=\rho}\right\} \frac{\psi_s(R(\tau))\,R'(\tau)\,f_M(\tau)}{R(\tau)-z}\,d\tau.$$

This goes to zero as $\rho \to 0$.
For, ψ_s belongs to M_c so that the second term vanishes and the first term vanishes since f_M belongs to M_p. Note that in this computation, z is taken from the Fatou set and the integration path γ_F runs near the Julia set. Note that this argument cannot be applied if $\rho = 0$ since the integrand might have a singularity at p which not regular. We need regularity of the singular points to have this kind of integral vanish. Finally,

$$(\mathcal{L}_{MM} f_M)(z) = \frac{1}{2\pi i}\int_{\gamma_M} \frac{\psi_s(R(\tau))\,R'(\tau)\,f_M(\tau)}{R(\tau)-z}\,d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_0^+ + \gamma_0^-} \frac{\psi_s(R(\tau))\,R'(\tau)\,f_M(\tau)}{R(\tau)-z}\,d\tau + \frac{1}{2\pi i}\int_{\gamma_p} \frac{\psi_s(R(\tau))\,R'(\tau)\,f_M(\tau)}{R(\tau)-z}\,d\tau$$

$$= I_c[(\Delta_c\psi_s)\cdot R_* f_M] + I_{R(p)}[\psi_s \cdot (\Delta_p f_M)\circ Q_a]$$

$$= [\psi_s \cdot R_* f_M]_c + [\psi_s \cdot f_M \circ Q_p]_{R(p)}.$$ This completes the proof of Proposition 5.2.

<u>Remark</u>. If $f_M \in M_0$, then the integrand may have a non-regular singularity, since we have a product of two regular singularities at $p=0$. Hence.

$$\mathcal{L}_{FM} f_M = [\psi_s \cdot R_* f_M]_F$$
$$\mathcal{L}_{MM} f_M = [\psi_s \cdot R_* f_M]_c$$
$$\mathcal{L}_{JM} f_M = [\psi_s \cdot R_* f_M]_J.$$

For $f_F \in \mathcal{O}_0(F)$, we have the following

<u>Proposition</u> 5.3

$$\mathcal{L}_{JF} f_F = 0$$
$$\mathcal{L}_{MF} f_F = [\psi_s \cdot R_* f_F]_c$$
$$\mathcal{L}_{FF} f_F = \psi_s \cdot R_* f_F - [\psi_s \cdot R_* f_F]_c$$

Proof As $f_F \in \mathcal{O}_0(F)$, we have an estimate $|f_F(\tau)| < k|\tau|^{-1}$ near ∞. Hence

$$\left|(\mathcal{L}_{JF} f_F)(z)\right| \leqslant \frac{2\pi|\tau|}{2\pi} \frac{|2\tau|^{-s}\,|2\tau|\cdot k|\tau|^{-1}}{|\tau^2|\,|1+\frac{c}{\tau^2}+\frac{z}{\tau^2}|} \longrightarrow 0 \quad (\text{as } \tau \to \infty)$$

Therefore we have $\mathcal{L}_{JF} f_F = 0$. For the second component,

$$\mathcal{L}_{MF} f_F = I_c[(\Delta_c \Psi_s)\cdot R_* f_F] = [\Psi_s \cdot R_* f_F]_c .$$

And the third component is computed similarly.

$$(\mathcal{L}_{FF} f_F)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\Psi_s(R(\tau))\,R'(\tau)\,f_F(\tau)}{R(\tau)-z}\,d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_F} \frac{\Psi_s(\sigma)\,(R_* f_F)(\sigma)}{\sigma - z}\,d\sigma$$

$$= \Psi_s(z)\cdot(R_* f_F)(z) - [\Psi_s \cdot R_* f_F]_c .$$

Putting the above propositions together, we have the dcomposition of our transfer operator into components.

$$\begin{pmatrix} \mathcal{L}_{JJ} & \mathcal{L}_{JM} & \mathcal{L}_{JF} \\ \mathcal{L}_{MJ} & \mathcal{L}_{MM} & \mathcal{L}_{MF} \\ \mathcal{L}_{FJ} & \mathcal{L}_{FM} & \mathcal{L}_{FF} \end{pmatrix}\begin{pmatrix} f_J \\ f_M \\ f_F \end{pmatrix} = \begin{pmatrix} \Psi_s\cdot R_* f_J - [\Psi_s \cdot R_* f_J]_c & [\Psi_s\cdot R_* f_M]_J & 0 \\ [\Psi_s \cdot R_* f_J]_c & [\Psi_s \cdot R_* f_M]_c + [\Psi_s\cdot f_M \circ Q_p]_{R(p)} & [\Psi_s \cdot R_* f_F]_c \\ 0 & [\Psi_s \cdot R_* f_M]_F & \Psi_s \cdot R_* f - [\Psi_s \cdot R_* f_F]_c \end{pmatrix}$$

Note that if f_J or f_F are not identically zero, then $[\Psi_s \cdot R_* f_J]_c \neq 0$ or $[\Psi_s \cdot R_* f_F]_c \neq 0$. Hence, there is no eigenfunction in subspace $\mathcal{O}(\mathbb{C}) \oplus \mathcal{O}_0(F)$.

§6. Invariant subspace of the transfer operator

Our transfer operator $\mathcal{L}_s : \mathcal{M}_+ \longrightarrow \mathcal{M}_+$ maps the space of pre-microfunctions supported on the forward orbit of the critical point. For $h_0 \in \mathcal{M}_c$, $h_0(z)$ can be written in a form of Cauchy integral.

$$h_0(z) = \frac{-1}{2\pi i}\int_{\gamma_c} (\Delta_c h_0)(t)\,\chi_t(z)\,dt = \frac{1}{2\pi i}\int_{\gamma_c} (\Delta_c h_0)(t)\,\chi_z(t)\,dt.$$

In the following, we denote as $\Psi_{s,k}(z) = \Psi_s(R_k(z))\cdot\Psi_s(R_{k-1}(z))\cdots\Psi_s(R(z))$

Where $R_k(z) = R \circ R_{k-1}(z)$ denote the k-th iteration of $R(z)$. Note that $\Psi_s \circ R_k \in \mathcal{M}_{R^{-k+1}(c)}$ and that $\Psi_{s,k}$ are regular on $\gamma_{P(R)}$.

For each $k = 1, 2, \cdots$, we consider a pre-microfunction h_k in $\mathcal{M}_{R_k(c)}$ expressed in terms of a pre-microfunction g_k in $\mathcal{M}_c$. For $g_k \in \mathcal{M}_c$, let

$$G_k(\zeta) = g_k(\zeta) \cdot \Psi_{s,k}(\zeta)$$

and define $h_k \in \mathcal{M}_{R_k(c)}$ by

$$h_k(z) = \frac{-1}{2\pi i} \int_{\gamma_c} (\Delta_c G_k)(t)\, R_k'(t)\, \chi_{R_k(t)}(z)\, dt.$$

Let $Q_k = (R_k|_{\gamma_c})^{-1} : \gamma_{R_k(c)} \longrightarrow \gamma_c$ be the inverse branch of R_k. By a change of variables $\sigma = R_k(t)$, $d\sigma = R_k'(t)\,dt$ and $t = Q_k(\sigma)$, we have

$$h_k(z) = \frac{-1}{2\pi i} \int_{\gamma_{R_k(c)}} \left(\Delta_{R_k(c)}[G_k \circ Q_k]\right)(\sigma)\, \chi_\sigma(z)\, d\sigma$$

This implies $h_k = [G_k \circ Q_k]_{R_k(c)}$ and $h_k \in \mathcal{M}_{R_k(c)}$.

We have
$$\Delta_c G_k = (\Delta_{R_k(c)} h_k) \circ R_k \quad \text{along } \gamma_c.$$

So, this correspondence induces an isomorphism $\mathcal{M}_{R_k(c)} \simeq \mathcal{M}_c$. As $h_k \in \mathcal{M}_{R_k(c)}$, we have $\mathcal{L}_s h_k \in \mathcal{M}_{R_{k+1}(c)} \oplus \mathcal{M}_c$. More precisely, we have the following explicit formula.

<u>Proposition</u> 6.1 If $h_k = [G_k \circ Q_k]_{R_k} \in \mathcal{M}_{R_k(c)}$ with G as above, we have the following decomposition.

$$\mathcal{L}_s h_k = [G_k \circ Q_{k+1} \cdot \Psi_s]_{R_{k+1}(c)} + [\Psi_s \cdot [G_k \circ Q_{k+1}]_{R_{k+1}(c)}]_c$$

<u>Proof</u> This is immediately verified by applying Theorem 4.4. By an immediate calculation, we can obtain the proof as follows. First component of $\mathcal{L}_s h_k$ is given by,

$$[\mathcal{L}_s h_k]_{R_{k+1}(c)} = \left[\mathcal{L}_s\left[\frac{-1}{2\pi i} \int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\, \chi_\sigma(z)\, d\sigma\right]\right]_{R_{k+1}(c)}$$

$$= \left[\frac{-1}{2\pi i} \int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\, \mathcal{L}_s[\chi_\sigma]\, d\sigma\right]_{R_{k+1}(c)}$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\left[\mathcal{L}_s \chi_\sigma\right]_{R_{k+1}(c)} d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\cdot \psi_s(R(\sigma)) R'(\sigma)\chi_{R(\sigma)}\, d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)}[G_k \circ Q_{k+1}])(\rho)\psi_s(\rho)\,\chi_\rho\, d\rho$$

$$= [\psi_s \cdot G_k \circ Q_{k+1}]_{R_{k+1}(c)},$$

where we made change of variables $\rho = R(\sigma)$, $d\rho = R'(\sigma)d\sigma$. Similarly, the second component is computed as follows.

$$[\mathcal{L}_s h_k]_c = \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)[\mathcal{L}_s \chi_\sigma]_c\, d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\cdot R'(\sigma)[\psi_s \cdot \chi_{R(\sigma)}]_c\, d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)}[G_k \circ Q_{k+1}])(\rho)\cdot[\psi_s \cdot \chi_\rho]_c\, d\rho$$

$$= \left[\psi_s \cdot \frac{-1}{2\pi i}\int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)}[G_k \circ Q_{k+1}])(\rho)\chi_\rho\, d\sigma\right]_c$$

$$= [\psi_s \cdot [G_k \circ Q_{k+1}]_{R_{k+1}(c)}]_c .$$

§7 Eigenvalue problem

In this section, we consider the eigenvalue problem for our transfer operator $\mathcal{L}_s$ restricted to an invariant subspace of pre-microfunctions $\mathcal{M}_{P(R)}$ defined in section 4 as

$$\mathcal{M}_{P(R)} = \sum_{k=0}^{\infty} \mathcal{M}_{R_k(c)} .$$

Here, the sum is considered as formal sum. In the case of postcritically finite maps, the post critical set $P(R)$ is a finite set and the sum is finite. In this case

$$\mathcal{M}_{P(R)} = \bigoplus_{p \in P(R)} \mathcal{M}_p .$$

In order to analize the eigenvalue problem of $\mathcal{L}_s$, we consider a formal sum of pre-microfunctions.

$$h = \sum_{k=0}^{\infty} h_k \quad \text{with} \quad h_k \in \mathcal{M}_{R_k(c)}$$

<u>Proposition</u> 7.1. If h is an eigenfunction of $\mathcal{L}_s$ satisfying

$$\lambda \mathcal{L}_s h = h \quad \text{and} \quad E(R) \text{ is infinite,}$$

then $\lambda [\mathcal{L}_s h_k]_{R_{k+1}(c)} = h_{k+1}$ and $\lambda \sum_{k=0}^{\infty} [\mathcal{L}_s h_k]_c = h_0$.

~~Proof~~ By a straightforward calculation, we have

$$\mathcal{L}_s h_k = [\mathcal{L}_s h_k]_{R_{k+1}(c)} + [\mathcal{L}_s h_k]_c .$$

<u>Theorem</u> 7.2 The eigenvalue problem $\lambda \mathcal{L}_s h = h$ of our transfer operator $\mathcal{L}_s : \mathcal{M}_{E(R)} \to \mathcal{M}_{E(R)}$ reduces to an "eigenvalue" problem of an integral operator

$$T_s : \mathcal{O}_0(\gamma_c) \longrightarrow \mathcal{O}_0(\gamma_c)$$

defined by

$$(T_s[\varphi])(u) = (\Delta_c \psi_s)(u) \cdot \frac{1}{2\pi i} \int_{\gamma_c} H_s(u, t ; \lambda)\, \varphi(t)\, dt$$

where

$$H_s(u,t;\lambda) = -\sum_{k=0}^{\infty} \lambda^k \psi_{s,k}(t)\, R'_{k+1}(t)\, \chi_{R_{k+1}(t)}(u),$$

with $\lambda T_s h_0 = h_0$, $h_0 = \Delta_c h$.

~~Proof~~ As $h_{k+1} = \lambda [\mathcal{L}_s h_k]_{R_{k+1}(c)} = \lambda [\psi_s \cdot G_k \circ Q_{k+1}]_{R_{k+1}(c)}$

$= \lambda [g_k \circ Q_{k+1} \cdot \psi_s \circ R_{k+1} \circ Q_{k+1} \cdot \psi_{s,k} \circ Q_{k+1}]_{R_{k+1}(c)}$

$= \lambda [g_k \circ Q_{k+1} \cdot \psi_{s,k+1} \circ Q_{k+1}]_{R_{k+1}(c)}$

and $h_{k+1} = [G_{k+1} \circ Q_{k+1}]_{R_{k+1}(c)} = [g_{k+1} \circ Q_{k+1} \cdot \psi_{s,k+1} \circ Q_{k+1}]_{R_{k+1}(c)}$

we have $g_{k+1} = \lambda g_k$ for $k \geq 0$.

Hence $g_k = \lambda^k h_0$, which implies

$$h_k = [G_k \circ Q_k]_{R_k(c)} = [\lambda^k h_0 \circ Q_k \cdot \psi_{s,k} \circ Q_k]_{R_k(c)}$$

and $h_0 = \lambda \sum_{k=0}^{\infty} [\mathcal{L}_s h_k]_c = \lambda \sum_{k=0}^{\infty} [\psi_s \cdot [G_k \circ Q_{k+1}]_{R_{k+1}(\cdot)}]_c$

$= \lambda \sum_{k=0}^{\infty} [\psi_s \cdot \lambda^k [h_0 \circ Q_{k+1} \cdot \psi_{s,k} \circ Q_{k+1}]_{R_{k+1}(\cdot)}]_c$

$= \lambda \sum_{k=0}^{\infty} \Big[\psi_s \cdot \lambda^k \frac{-1}{2\pi i} \int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)} [h_0 \circ Q_{k+1}])(\sigma) \cdot \psi_{s,k} \circ Q_{k+1}(\sigma) \chi_\sigma \, d\sigma \Big]_c$

$= \lambda \sum_{k=0}^{\infty} \Big[\psi_s \cdot \lambda^k \frac{-1}{2\pi i} \int_{\gamma_c} (\Delta_c h_0)(\tau) \psi_{s,k}(\tau) \cdot R'_{k+1}(\tau) \chi_{R_{k+1}(\tau)} \, d\tau \Big]_c$

(here we changed variables $\tau = Q_{k+1}(\sigma)$ and $d\sigma = R'_{k+1}(\tau) d\tau$)

$= \frac{-\lambda}{2\pi i} \int_{\gamma_c} (\Delta_c \psi_s)(u) \Big(\frac{-1}{2\pi i} \int_{\gamma_c} \sum_{k=0}^{\infty} \lambda^k \psi_{s,k}(t) \cdot R'_{k+1}(t) \chi_{R_{k+1}(t)}(u) \, (\Delta h_0)(t) \, dt \Big) \chi_u \, du$

This yields an integral equation for $h_0 \in \mathcal{M}_c$:

$$(\Delta h_0)(u) = \lambda (\Delta_c \psi_s)(u) \frac{1}{2\pi i} \int_{\gamma_c} H_s(u,t;\lambda) (\Delta_c h_0)(t) \, dt$$

more biefly

$$h_0 = \lambda \Big[\psi_s \cdot \frac{1}{2\pi i} \int_{\gamma_c} H_s(u,t;\lambda) (\Delta_c h_0)(t) \, dt \Big]_c,$$

by setting $H_s(u,t;\lambda) = -\sum_{k=0}^{\infty} \lambda^k \psi_{s,k}(t) R'_{k+1}(t) \chi_{R_{k+1}(t)}(u)$.

§8. Dual spaces and Cauchy's transformations.

Let $\mathcal{O}(J)$ denote the space of germs of holomorphic function along the Julia set $J = J(R)$. Each element of $\mathcal{O}(J)$ has a reprecentative $f: J \to \mathbb{C}$ which can be extended to a holomorphic function in a neighborhood of J. As J is a perfect set, the analytic continuation is uniquely determined by f. Topology in $\mathcal{O}(J)$ is given by the uniform convergence on J.

Linear functional $G^J: \mathcal{O}(J) \to \mathbb{C}$ is said to be holomorphic if for any holomorphic family $f_\lambda: \Lambda \to \mathcal{O}(J)$, $G^J[f_\lambda]: \Lambda \to \mathbb{C}$ is holomorphic. We require the continuity of G^J with respect to the sup norms on $N_\varepsilon(J)$. The space of continuous holomorphic linear functionals $G^J: \mathcal{O}(J) \to \mathbb{C}$ will be denoted by $\mathcal{O}^*(J)$. As in the previous sections, $\mathcal{O}_0(F)$ denotes the space of holomorphic functions in the Fatou set F vanishing at the infinity. If $p \in F$ then $\chi_p = \frac{1}{z-p}$ belongs to $\mathcal{O}(J)$. For holomorphic linear functional G^J in $\mathcal{O}^*(J)$, define a holomorphic function $g^J \in \mathcal{O}_0(F)$ by $g^J(z) = G^J[-\chi_z]$. Then family

of holomorphic functions $F \to \mathcal{O}(J)$ defined by $z \mapsto -\chi_z$ is a holomorphic family, $g^J : F \to \mathbb{C}$ is a holomorphic function, since the functional G^J is holomorphic. By the continuity of G^J, we see immediately $g^J(\infty) = 0$ and hence $g^J \in \mathcal{O}_0(F)$.
This correspondence between $\mathcal{O}^*(J)$ and $\mathcal{O}_0(F)$ is called the Cauchy transformation.

Proposition 8.1. For $f_J \in \mathcal{O}(J)$, $G^J[f_J]$ can be expressed in a integration form

$$G^J[f_J] = \frac{1}{2\pi i} \int_{\gamma_F} f_J(\tau)\, g^J(\tau)\, d\tau,$$

where the integration path γ_F goes around the Julia set in the clockwise direction.

Proof As f_J is holomorphic near J, for z in a neighborhood of J,

$$f_J(z) = \frac{1}{2\pi i} \int_{\gamma_J} f_J(\tau) \chi_z(\tau)\, d\tau = \frac{-1}{2\pi i} \int_{\gamma_J} f_J(\tau) \chi_\tau(z)\, d\tau,$$

where γ_J runs around the Julia set in the counter-clockwise direction. The right hand side of this equality gives an expression of $f_J(z)$ as a "linear combination" of unit poles. We have

$$G^J[f_J] = \frac{-1}{2\pi i} \int_{\gamma_J} f_J(\tau)\, G^J[\chi_\tau]\, d\tau = \frac{-1}{2\pi i} \int_{\gamma_J} f_J(\tau)\, g^J(\tau)\, d\tau$$

$$= \frac{1}{2\pi i} \int_{\gamma_F} f_J(\tau)\, g^J(\tau)\, d\tau.$$

In the following, we shall denote such pairings of functions as

$$\langle g^J, f_J \rangle_F = \frac{1}{2\pi i} \int_{\gamma_F} f_J(\tau)\, g^J(\tau)\, d\tau.$$

Proposition 8.2 $\quad \mathcal{O}^*(J) \approx \mathcal{O}_0(F)$

Proof The Cauchy transformation defines a complex linear map from $\mathcal{O}^*(J)$ to $\mathcal{O}_0(F)$, and the pairing along γ_F defines a complex linear map from $\mathcal{O}_0(F)$ to $\mathcal{O}^*(J)$. These two transformation are mutually inverse.

Let $\mathcal{O}_0^*(F)$ denote the space of holomorphic linear and continuous functional $G^F : \mathcal{O}_0(F) \to \mathbb{C}$.

Proposition 8.3 $\quad \mathcal{O}(J) \subset \mathcal{O}_0^*(F)$

Proof. If $z \in J$ then $-\chi_z \in \mathcal{O}_0(F)$. For $G^F \in \mathcal{O}_0^*(F)$, let $g^F(z) = G^F[-\chi_z]$. Then $g^F : J \to \mathbb{C}$ is a continuous function.

If $g^F \in \mathcal{O}(J)$, then for $f_F \in \mathcal{O}_0(F)$ with

$$f_F(z) = \frac{1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,\chi_z(\tau)\,d\tau = \frac{-1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,\chi_\tau(z)\,d\tau,$$

we have

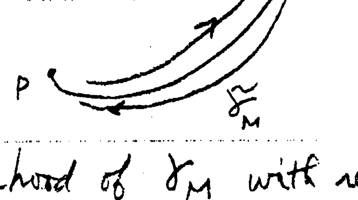
$$G^F[\,f_F\,] = \frac{-1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,G^F[\chi_\tau]\,d\tau = \frac{-1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,g^F(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_J} f_F(\tau)\,g^F(\tau)\,d\tau = \langle g^F, f_F \rangle_J,$$

where the integration path γ_J is given by g^F which is holomorphic in a neighborhood of J. This pairing will be denoted as $\langle g^F, f_F \rangle_J$.

We define the third pairings $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{\tilde{M}}$ related to the external rays and pre-microfunctions. Let $\mathcal{M}$ denote the space of pre-microfunctions and let γ_M denote the "sum" of external rays supporting the pre-microfunctions. We use symbol M to indicate the object is related to the pre-microfunction component. When we apply operations Δ_M, I_M etc. we take the "sum" of the objects over external rays. The dual space $\mathcal{M}^*$ is the space of holomorphic linear and continuous functionals $G^M : \mathcal{M} \to \mathbb{C}$.

For γ_M, we denote by $\tilde{\gamma}_M$ the integration path passing along the external ray both sides of γ_M coming from the infinity to p on the negative side of γ_M and coming back from p to the infinity on the positive side of γ_M.



If $g^M \in \mathcal{O}_0(\gamma_M)$, that is, g^M is a holomorphic function in a neighborhood of γ_M with regular singularities at the infinity and each landing points.

For $f_M \in \mathcal{M}$, we can rewrite it in the following form.

$$f_M(z) = \frac{1}{2\pi i}\int_{\gamma_M} (\Delta_M f_M)(\tau)\chi_z(\tau)\,d\tau = \frac{-1}{2\pi i}\int_{\gamma_M} (\Delta_M f_M)(\tau)\chi_\tau(z)\,d\tau$$

For $g^M \in \mathcal{O}(\gamma_M)$ define a holomorphic functional $G^M \in \mathcal{M}^*$ by

$$G^M[f_M] = \langle g^M, \Delta_M f_M \rangle_M = \langle g^M, f_M \rangle_{\tilde{M}}.$$

$G^M[f_M]$ is defined if $\Delta_M f_M \cdot g^M \in L_1(\gamma_M)$. Note that

$$G^M[-\chi_z] = \langle g^M, -\chi_z \rangle_{\tilde{M}} = g^M(z)$$

by Cauchy's integration formula. Note that if $g^M \in \mathcal{O}_0(\gamma_M)$ and $\tilde{g}^M = I_M[g^M]$, then for $\zeta \in \mathbb{C}\setminus\gamma_M$

$$\tilde{g}^M(\zeta) = \langle g^M, \chi_\zeta \rangle_{\tilde{M}} = \langle g^M, \chi_\zeta \rangle_M$$

holds. If $f_M \in \mathcal{M}$, then

$$G^M[f_M] = G^M\Big[\frac{1}{2\pi i}\int_{\gamma_M}(\Delta_M f_M)(\tau)(-\chi_\tau)\cdot d\tau\Big] = \frac{1}{2\pi i}\int_{\gamma_M}(\Delta_M f_M)(\tau)\,G^M[-\chi_\tau]\,d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_M}(\Delta_M f_M)(\tau)\,g^M(\tau)\,d\tau = \langle g^M, \Delta_M f_M \rangle_M.$$

We have a splitting of pre-microfunctions,

$$f = f_J \oplus f_M \oplus f_F \in \mathcal{O}_0(J) \oplus \mathcal{M} \oplus \mathcal{O}_0(F)$$

and a splitting of its dual space

$$G = G^J \oplus G^M \oplus G^F \in \mathcal{O}_0^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}_0^*(F),$$

with Cauchy's transforms given by

$$\begin{aligned}
G^J[-\chi_z] &= g^J(z), && z \in F, \quad g^J \in \mathcal{O}_0(F)\\
G^M[-\chi_z] &= g^M(z), && z \in \gamma_M, \quad g^M \in \mathcal{O}(\gamma_M)\\
G^M[\chi_\zeta] &= \tilde{g}^M(\zeta), && \zeta \in \mathbb{C}\setminus\gamma_M, \quad \tilde{g}^M \in \widehat{\mathcal{M}}\\
G^F[-\chi_z] &= g^F(z), && z \in J, \quad g^F \in \mathcal{O}(J).
\end{aligned}$$

The pairing of f an G is defined by

$$\begin{aligned}
G[f] &= G^J[f_J] + G^M[f_M] + G^F[f_F]\\
&= \langle g^J, f_J \rangle_J + \langle g^M, \Delta_M f_M \rangle_M + \langle g^F, f_F \rangle_F.
\end{aligned}$$

Projections of $\mathcal{H} = \mathcal{O}_0(J) \oplus \mathcal{M} \oplus \mathcal{O}_0(F)$ to components are denoted by $f \mapsto [f]_J$, $f \mapsto [f]_M$, $f \mapsto [f]_F$ respectively.

These projections are given by

$$[f]_J(z) = \frac{1}{2\pi i}\int_{\gamma_J} f(\tau)\chi_z(\tau)\,d\tau \qquad (z \in J),$$

$$[f]_M(z) = \frac{1}{2\pi i}\int_{\gamma_M} (\Delta_M f)(\tau)\chi_z(\tau)\,d\tau \qquad (z \in \mathbb{C}\setminus\gamma_M),$$

$$[f]_F(z) = \frac{1}{2\pi i}\int_{\gamma_F} f(\tau)\chi_z(\tau)\,d\tau \qquad (z \in F).$$

And the projections of $\mathcal{H}^* = \mathcal{O}^*(J)\oplus\mathcal{M}^*\oplus\mathcal{O}_0^*(F)$ are denoted as

$$[\ ]^J : \mathcal{H}^* \longrightarrow \mathcal{O}_0(F) \subset \mathcal{O}^*(J)$$

$$[\ ]^M : \mathcal{H}^* \to \mathcal{O}(\gamma_M) \subset \mathcal{M}^*$$

$$[\ ]^F : \mathcal{H}^* \longrightarrow \mathcal{O}(J) \subset \mathcal{O}_0^*(F).$$

Let $\mathcal{L}_s^*$ denote the dual of our transfer operator $\mathcal{L}_s$. We abuse notations and confuse functionals and its Cauchy's transforms. $\mathcal{L}_s^* : \mathcal{O}_0(F)\oplus\mathcal{O}(\gamma_M)\oplus\mathcal{O}(J) \circlearrowleft$ is decomposed as

$$\mathcal{L}_s^* = \begin{pmatrix} \mathcal{L}_{JJ}^* & \mathcal{L}_{JM}^* & \mathcal{L}_{JF}^* \\ \mathcal{L}_{JM}^* & \mathcal{L}_{MM}^* & \mathcal{L}_{MF}^* \\ \mathcal{L}_{FJ}^* & \mathcal{L}_{FM}^* & \mathcal{L}_{FF}^* \end{pmatrix}.$$

In the rest of this section, we compute these components more explicitly.

<u>Proposition</u> 8.4.

$$\mathcal{L}_{JJ}^* g^J = \psi_s\circ R\cdot R'\cdot g^J\circ R - [\psi_s\circ R\cdot R'\cdot g^J\circ R]_0$$

$$\mathcal{L}_{MJ}^* g^J = \Delta_0[\psi_s\circ R\cdot R'\cdot g^J\circ R]$$

$$\mathcal{L}_{FJ}^* g^J = 0$$

<u>Proof</u>. For $g^J \in \mathcal{O}_0(F)$, we compute $(\mathcal{L}_{JJ}^* g^J)(\zeta)$ for $\zeta\in F$.

$$(\mathcal{L}_{JJ}^* g^J)(\zeta) = [(\mathcal{L}_s^* G^J)[-\chi_\zeta]]^J = [G^J[-\mathcal{L}_s\chi_\zeta]]^J$$

$$= [G^J[-\psi_s(R(\zeta))\cdot R'(\zeta)\,\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^J$$

$$= [G^J[-\psi_s(R(\zeta))\cdot R'(\zeta)\chi_{R(\zeta)}]]^J \qquad (\text{since } [\ ]_c \in \mathcal{M}_c)$$

$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\,G^J[-\chi_{R(\zeta)}]]^J = [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^J[-\chi_{R(\zeta)}]]_F$$

$$= \psi_s(R(\zeta))\,R'(\zeta)\,g^J(R(\zeta)) - [\psi_s\circ R\cdot R'\cdot g^J\circ R]_0(\zeta).$$

Next, for $\zeta \in \gamma_M$, $\mathcal{L}^*_{MJ}\, g^J \in \mathcal{O}(\gamma_M)$ is computed as follows.

$$(\mathcal{L}^*_{MJ}\, g^J)(\zeta) = [(\mathcal{L}^*_s G^J)[-\chi_\zeta]]^M$$
$$= [G^J[-\psi_s(R(\zeta))\cdot R'(\zeta)\ \chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot \chi_{R(\zeta)}]_c]]^M$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\ G^J[-\chi_{R(\zeta)}]]^M = [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^J(R(\zeta))]^M$$
$$= \Delta_0[\psi_s\circ R\cdot R'\cdot g^J\circ R](\zeta) = \Delta_c\psi_s\circ R\cdot R'\cdot g^J\circ R.$$

For $\zeta \in J$, we have

$$(\mathcal{L}^*_{FJ}\, g^J)(\zeta) = [(\mathcal{L}^*_s G^J)[-\chi_\zeta]]^F$$
$$= [G^J[-\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^M$$
$$= 0$$

The last equality holds since $\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} \in \mathcal{O}_0(F)$ and $[\psi_s\cdot R(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$.

<u>Proposition</u> 8.5 $\quad \mathcal{L}^*_{JF}\, g^F = 0$

$$\mathcal{L}^*_{MF}\, g^F = \Delta_0[\psi_s\circ R\cdot R'\cdot g^F\circ R]$$
$$\mathcal{L}^*_{FF}\, g^F = \psi_s\circ R\cdot R'\cdot g^F\circ R - [\psi_s\circ R\cdot R'\cdot g^F\circ R]_0$$

<u>proof</u>. For $\zeta\in F$, we have $-\chi_\zeta \in \mathcal{O}(J)$. For $g^F \in \mathcal{O}(J)$,

$$(\mathcal{L}^*_{JF}\, g^F)(\zeta) = [(\mathcal{L}^*_s G^F)[-\chi_\zeta]]^J = [G^F[-\mathcal{L}_s\chi_\zeta]]^J$$
$$= [G^F[-\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^J$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^F[-\chi_{R(\zeta)}]]^J = 0.$$

Here, the last equality holds since $\psi_s(R(\zeta))\cdot R'(\zeta)\,\chi_{R(\zeta)} \in \mathcal{O}(J)$, $[\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$ and $G^F[-\chi_{R(\zeta)}] = 0$.

For $\zeta \in J$, $-\chi_\zeta$ belongs to $\mathcal{O}_0(F)$. Hence

$$(\mathcal{L}^*_{FF}\, g^F)(\zeta) = [(\mathcal{L}^*_s G^F)[-\chi_\zeta]]^F = [G^F[-\mathcal{L}_s\chi_\zeta]]^F$$
$$= [G^F[-\psi_s(R(\zeta))\cdot R'(\zeta)\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^F$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^F[-\chi_{R(\zeta)}]]^F = [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^F(R(\zeta))]^F$$
$$= \psi_s\circ R(\zeta)\cdot R'(\zeta)\cdot g^F\circ R(\zeta) - [\psi_s\circ R\cdot R'\cdot g^F\circ R]_0(\zeta).$$

We used the fact $-\psi_s(R(\zeta))\cdot R'(\zeta)\chi_{R(\zeta)} \in \mathcal{O}_0(F)$ and $[\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$.

To compute $\mathcal{L}^*_{MF} g^F$, we take $\zeta \in \gamma_M \subset F$. Then,

$$(\mathcal{L}^*_{MF} g^F)(\zeta) = [(\mathcal{L}^*_s G^F)[-\chi_\zeta]]^M$$
$$= [G^F[-\psi_s(R(\zeta)) \cdot R'(\zeta)\chi_{R(\zeta)} - [\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c]]^M$$
$$= [\psi_s(R(\zeta)) \cdot R'(\zeta) \cdot G^F[-\chi_{R(\zeta)}]]^M = [\psi_s(R(\zeta)) \cdot R'(\zeta) \cdot g^F(R(\zeta))]^M$$
$$= \Delta_0[\psi_s \circ R \cdot R' \cdot g^F \circ R](\zeta).$$

In the above calculations, we used the fact $[\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c \in \mathcal{M}_c$. During the computations, ζ is regarded as constant and the final result gives the formula as a function of ζ.

<u>Proposition</u> 8.6. $\mathcal{L}^*_{JM} g^M = [\psi_s \circ R \cdot R' \cdot (I_M g^M) \circ R]_F$

$$\mathcal{L}^*_{MM} g^M = \begin{cases} [\psi_s \circ R \cdot R' \cdot g^M \circ R]^{M_p} + [I_0[g^M \circ R \cdot (\Delta_0 \psi_s) \circ R \cdot R']]^{M_p} & (\zeta \in \gamma_p \text{ and } p \neq 0) \\ \left[\frac{1}{2\pi i}\int_{\gamma_0} g^M(R(\sigma))\psi_s(R(\sigma)) \cdot R'(\sigma)\chi_\zeta(\sigma)\,d\sigma\right]^{M_0} & (\zeta \in \gamma_0) \end{cases}$$

$$\mathcal{L}^*_{FM} g^M = [\psi_s \circ R \cdot R' \cdot (I_M g^M) \circ R]_J$$

<u>Proof</u> For $G^M \in \mathcal{M}^*$, Let $g^M(z) = G^M[-\chi_z]$, $z \in M$, $g^M \in \mathcal{O}(M)$. For $\zeta \in F$, (and $\zeta \in \mathbb{C} \setminus \gamma_M$),

$$(\mathcal{L}^*_{JM} G^M)(\zeta) = [(\mathcal{L}^* G^M)[\chi_\zeta]]^J = [G^M[\mathcal{L}_s \chi_\zeta]]^J$$
$$= [G^M[-\psi_s(R(\zeta)) \cdot R'(\zeta) \cdot \chi_{R(\zeta)} + [\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c]]^J$$
$$= \left[\psi_s(R(\zeta)) \cdot R'(\zeta)\, G^M[\chi_{R(\zeta)}] + \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c \psi_s)(\tau) R'(\zeta)\chi_{R(\zeta)}(\tau)\,d\tau\right]^J$$
$$= [\psi_s \circ R \cdot R' \cdot (I_M g^M) \circ R]_F(\zeta) + \frac{1}{2\pi i}\int_{\gamma_F} \frac{dz}{z-\zeta} \cdot \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau) \cdot R'(z)\chi_{R(z)}(\tau)\,d\tau$$
$$= [\psi_s \circ R \cdot R' \cdot (I_M g^M) \circ R]_F(\zeta) + \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)\,d\tau \cdot \frac{1}{2\pi i}\int_{\gamma_F} \frac{R'(z)\,dz}{(z-\zeta)(\tau - R(z))}$$
$$= [\psi_s \circ R \cdot R' \cdot (I_M g^M) \circ R]_F(\zeta).$$

Here, the last equality holds since $\zeta \in F \cap (\mathbb{C} \setminus \gamma_M)$ and z moves near ∂F along γ_F.

Next, we compute $\mathcal{L}^*_{FM}$. For $\zeta \in J$, note that $R(\zeta) \in J$ and $[\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c \in \mathcal{M}$. Hence,

$$(\mathcal{L}^*_{FM}G^M)(\zeta) = [(\mathcal{L}^*G^M)[\chi_\zeta]]^F$$
$$= [G^M[\psi_s(R(\zeta))\cdot R'(\zeta)\cdot \chi_{R(\zeta)} + [\psi_s \cdot R'(\zeta)\cdot \chi_{R(\zeta)}]_c]]^F$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot (I_M g^M)(R(\zeta))]^F + [\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\cdot (\Delta_c\psi_s)(\tau)\cdot R'(\zeta)\cdot \chi_{R(\zeta)}(\tau)d\tau]_J$$

The second term of the above line is computed as follows.

$$[\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\cdot(\Delta_c\psi_s)(\tau)\cdot R'(\zeta)\cdot\chi_{R(\zeta)}(\tau)d\tau]_J$$
$$= \frac{1}{2\pi i}\int_{\gamma_J}\frac{dz}{z-\zeta}\,\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\cdot(\Delta_c\psi_s)(\tau)\,R'(\tau)\,\frac{d\tau}{\tau - R(z)}$$
$$= \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\,(\Delta_c\psi_s)(\tau)\,d\tau\,\frac{1}{2\pi i}\int_{\gamma_J}\frac{R'(z)\,dz}{(z-\zeta)(\tau - R(z))}$$
$$= 0.$$

The last equality holds since $R'(z)$ is of degree one and the denominator $(z-\zeta)(\tau - R(z))$ is of degree three with respect to the variable of integration and the integration path γ_J turns around the Julia set along a circle of infinitely large radius. Hence we have

$$(\mathcal{L}^*_{FM}G^M)(\zeta) = [\psi_s(R(\zeta))\cdot R'(\zeta)(I_M g^M)(R(\zeta))]_J.$$

Finally, let us compute $\mathcal{L}^*_{MM}G^M \in \mathcal{O}^*(M)$.
For $\zeta \in \gamma_p$ with $p \in \mathcal{L}(R)$, the component $(\mathcal{L}^*_{MM}G^M)_p \in \mathcal{O}(\gamma_p)$ is computed as follows.

$$(\mathcal{L}^*_{MM}G^M)(\zeta) = [(\mathcal{L}^*_s G^M)[-\chi_\zeta]]^{M_p} = [G^M[-\mathcal{L}_s\chi_\zeta]]^{M_p}$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^M[-\chi_{R(\zeta)}] + \langle g^M, [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c\rangle_M]^{M_p}$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^M(R(\zeta))]^{M_p} + [\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)\cdot R'(\zeta)\chi_{R(\zeta)}(\tau)d\tau]^{M_p}$$
$$= [\psi_s\circ R\cdot R'\cdot g^M_{R(p)}\circ R]^{M_p}(\zeta) + [R'(\zeta)\cdot(I_c[g^M_c\cdot\Delta_c\psi_s])\circ R]^{M_p}$$
$$= [R'\cdot(\psi_s\cdot g^M_{R(p)})\circ R]^{M_p}(\zeta) + [R'\cdot[\psi_s\cdot g^M_c]_c\circ R]^{M_p}(\zeta).$$

In the case of $p = 0$, i.e., for $\zeta \in \gamma_0$, we have

$$(\mathcal{L}^*_{MM}G^M)(\zeta) = [(\mathcal{L}^*G^M)[-\chi_\zeta]]^{M_0} = [G^M[-\mathcal{L}_s\chi_\zeta]]^{M_p}$$
$$= [G^M[-\psi_s\cdot R'(\zeta)\chi_{R(\zeta)}]]^{M_0} = [\langle g^M, \psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}\rangle_{\gamma_c}]^{M_0}$$

$$= \left[\frac{1}{2\pi i} \int_{\tilde{\gamma}_c} g^M(\tau)\, \psi_s(\tau) \cdot R'(\zeta) \frac{d\tau}{\tau - R(\zeta)} \right]^{M_0}$$

$$= \left[\frac{1}{2\pi i} \int_{\gamma_0} g^M(R(\sigma))\, \psi_s(R(\sigma)) \cdot R'(\zeta) \chi_\zeta(\sigma)\, d\sigma \right]^{M_0}$$

$$= R'(\zeta) \cdot \left(\Delta_0 \left[[g_c^M \cdot \psi_s]_c \circ R \right] \right)(\zeta).$$

§9. Example

In this section, we compute the operator $\mathcal{L}^*_{MM}$ more precisely for $R(z) = z^2 + i$ case. In this case, the critical value $c = i$ is preperiodic and the postcritical set $P(R) = \{i, i-1, -i\}$ consists of three points.

Let us compute $\mathcal{L}^*_{MM} g_c^M$ for $g^M_c \in \mathcal{O}(\gamma_c)$, where $G_c^M \in \mathcal{M}_c^*$ and $g_c^M(\zeta) = G_c^M[-\chi_\zeta]$ for $\zeta \in \gamma_c$.

For $\zeta \in \gamma_p$ with $p \neq 0$, $p \in P(R)$

$$(\mathcal{L}^*_{MM} G^M)(\zeta) = \left[(\mathcal{L}^*_s G_c^M)[-\chi_\zeta] \right]^{M_p} = \left[G_c^M[-\mathcal{L}_s \chi_\zeta] \right]^{M_p}$$

$$= \left[G_c^M \left[-\psi_s(R(\zeta)) \cdot R'(\zeta)\, \chi_{R(\zeta)} - [\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c \right] \right]^{M_p}.$$

Here, as $p \neq 0$, $R(\zeta) \notin \gamma_c$, $\chi_{R(\zeta)} \notin \mathcal{M}_c$. And as $[\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c$ belongs to $\mathcal{M}_c$, we have

$$(\mathcal{L}^*_{MM} G^M)(\zeta) = \left[G_c^M \left[- [\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c \right] \right]^{M_p}$$

$$= \left[\frac{1}{2\pi i} \int_{\gamma_c} g_c^M(\tau) \cdot \Delta_c \psi_s)(\tau) \cdot R'(\zeta)\, \chi_{R(\zeta)}(\tau)\, d\tau \right]^{M_p}$$

$$= \left[R'(\zeta) \frac{1}{2\pi i} \int_{\gamma_c} g_c^M(\tau) (\Delta_c \psi_s)(\tau)\, \chi_{R(\zeta)}(\tau)\, d\tau \right]^{M_p}$$

$$= \left[R'(\zeta)\, I_c[g_c^M \cdot \Delta_c \psi_s] \circ R(\zeta) \right]^{M_p}$$

$$= \left(R' \cdot I_c[g_c^M \cdot \Delta_c \psi_s] \circ R \right)(\zeta).$$

For $\zeta \in \gamma_0$,

$$(\mathcal{L}^*_{MM} G_c^M)(\zeta) = \left[(\mathcal{L}^*_s G_c^M)[-\chi_\zeta] \right]^{M_0} = \left[G_c^M[-\mathcal{L}_s \chi_\zeta] \right]^{M_0}$$

$$= \left[G_c^M[-\psi_s \cdot R'(\zeta) \chi_{R(\zeta)}] \right]^{M_0} = \left[R'(\zeta) \frac{-1}{2\pi i} \int_{\tilde{\gamma}_c} g_c^M(\tau)\, \psi_s(\tau)\, \chi_{R(\zeta)}(\tau)\, d\tau \right]^{M_0}$$

$$= \left[R(\zeta) \frac{-1}{2\pi i} \int_{\tilde{\gamma}_c} g_c^M(\tau) \psi_s(\tau) \frac{d\tau}{\tau - R(\zeta)} \right]^{M_0}$$

$$= \left[R(\zeta) \{\psi_s \cdot g_c^M\}_c \circ R(\zeta) \right]^{M_0},$$

where $\{\psi_s \cdot g_c^M\}_c$ denotes the regular part of $\psi_c \cdot g_c^M$ along γ_c.

i.e. $$\{\psi_s \cdot g_c^M\}_c = \psi_s \cdot g_c^M - I_c[\Delta_c \psi_s \cdot g_c^M]$$

and is defined as

$$\{\psi_s \cdot g_c^M\}_c(\zeta) = \frac{1}{2\pi i} \int_{\tilde{\gamma}_c} \psi_s(\tau) \cdot g_c^M(\tau) \frac{d\tau}{\tau - \zeta} \quad \text{for } \zeta \in \gamma_c$$

We have a decomposition

$$\psi_s \cdot g_c^M = [\psi_s \cdot g_c^M]_c + \{\psi_s \cdot g_c^M\}_c$$

with $[\psi_s \cdot g_c^M]_c \in \mathcal{M}_c$ and $\{\psi_s \cdot g_c^M\}_c \in \mathcal{O}(\gamma_c)$.

§10. Complex conformal measures.

Let $G \in \mathcal{O}_0^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}_0^*(F)$. Let $A \subset \mathbb{C}$ be an open set with smooth boundary ∂A (oriented by the counter clockwise direction). The characteristic function $\chi_A(z)$ of A is expressed as

$$\chi_A(z) = \frac{-1}{2\pi i} \int_{\partial A} \chi_\eta(z)\, d\eta = \frac{-1}{2\pi i} \int_{\partial A} \frac{d\eta}{z - \eta}$$

So, we can rewrite

$$\chi_A = \frac{-1}{2\pi i} \int_{\partial A} \chi_\eta\, d\eta.$$

Hence,

$$G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} G[-\chi_\eta]\, d\eta$$

defines a set function. If $G = G^J + G^M + G^F$, then

$$G[-\chi_\eta] = g^J + g^M + g^F$$

with $g^J \in \mathcal{O}_0(F)$, $g^M \in \mathcal{M}$, $g^F \in \mathcal{O}_0(J)$, and

$$G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} (g^J(\eta) + g^M(\eta) + g^F(\eta))\, d\eta$$

defines an additive set function. Suppose λ be a characteristic value of our transfer operator $\mathcal{L}_s$ and let $f \in \mathcal{H} = \mathcal{O}_0(J) \oplus \mathcal{M} \oplus \mathcal{O}_0(F)$ be an eigenfunction of $\mathcal{L}_s$ for singular value λ, i.e. $\lambda \mathcal{L}_s f = f$. And let $G \in \mathcal{H}^* = \mathcal{O}_0^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}_0^*(F)$ be the co-eigenfunctional of $\mathcal{L}_s^*$ for λ, i.e., $\lambda \mathcal{L}_s^* G = G$, with $g(z) = G[-\chi_z]$, $g \in \mathcal{O}_0(F) \oplus \mathcal{M} \oplus \mathcal{O}_0(J)$.

Define a set function μ_{fg} by

$$\mu_{fg}(A) = \frac{1}{2\pi i}\int_{\partial A} f(\tau)\, g(\tau)\, d\tau .$$

Then, we have

$$\mu_{fg}(R^{-1}(A)) = \frac{1}{2\pi i}\int_{\partial R^{-1}(A)} f(z)\, g(z)\, dz$$

$$= \frac{1}{2\pi i}\int_{\partial R^{-1}(A)} f(z)\cdot \lambda(\mathcal{L}_s^* g)(z)\, dz$$

$$= \frac{1}{2\pi i}\int_{R^{-1}(\partial A)} \lambda f(z)\, R'(z)\, \Psi_s(R(z))\, g(R(z))\, dz$$

Then by a change of variables $\zeta = R(z)$ with $d\zeta = R'(z)dz$, we have

$$\mu_{fg}(R^{-1}(A)) = \frac{1}{2\pi i}\int_{\partial A} \lambda\, \Psi_s(\zeta)\cdot g(\zeta)\Big(\sum_{z\in R^{-1}(\zeta)} f(z)\Big)\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial A} \lambda(\mathcal{L}_s f)(\zeta)\cdot g(\zeta)\, d\zeta$$

$$= \mu_{fg}(A),$$

where we used $\mathcal{L}_s f = \Psi_s \cdot R_* f$ and $\mathcal{L}_s^* g = R'(\Psi_s \circ g)\circ R$. Our set function μ_{fg} is backward invariant under R.

Finally, we consider the pull-back of the set function defined by the co-eigenfunction g. Suppose $\mathcal{L}_s^* g = g$ then, for A with $R|_A : A \to R(A)$ injective, we have

$$\mu_g(R(A)) = \frac{1}{2\pi i}\int_{\partial R(A)} g(\zeta)\, d\zeta = \frac{1}{2\pi i}\int_{\partial A} g(R(z))\, R'(z)\, dz$$

$$= \frac{1}{2\pi i}\int_{\partial A} \Psi_s(R(z))^{-1}\cdot R'(z)\cdot \Psi_s(R(z))\cdot g(R(z))\, dz$$

$$= \frac{1}{2\pi i}\int_{\partial A} \Psi_s(R(z))^{-1}\, g(z)\, dz.$$

This shows a kind of complex conformal property of the set function μ_g for co-invariant function g.